Journal of Applied Mechanics and Technical Physics, Vol. 42, No. 4, pp. 621–627, 2001

LIFT FORCE OF AN AIRFOIL IN THE FORM OF AN ARC WITH A SINK

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UDC 532.5

The problem of maximizing the lift force of an airfoil in the form of an arc with a sink modeling flow removal is studied within the framework of the classical model of a steady ideal incompressible liquid flow. For a sink with a fixed flow rate, an optimal position on the upper surface of the arc is found, which ensures the greatest increase in the lift force. It is shown that, in the presence of a sink, the optimal shape of the arc with a limited curvature and chord length coincides with the optimal shape of the arc without a sink, which was designed by M. A. Lavrent'ev (an arc of a circle). The flow rate corresponding to the maximum lift force is determined, and the mechanism of the influence of flow removal on the lift force is examined.

Introduction. In studying complex problems of aerohydrodynamics, at the initial stage, it is reasonable to choose the simplest mathematical model of the flow and a simple shape of the airfoil. This was what Lavrent'ev did [1, 2] solving the extreme problem of the airfoil theory, where an airfoil in the form of an arc was considered within the framework of the ideal incompressible liquid model. In studying the effect of boundary-layer suction from the airfoil surface on the lift and drag forces, Golubev was one of the first researchers to replace a slot by a point sink [3, 4]. The paper of Nekrasov [5] on the flow around an airfoil with a sink and a source should be also noted. An extreme problem with a slot on the airfoil replaced by a sink was considered by Abzalilov and Il'inskii [6].

In the present work, we study the problem of maximizing the lift force of an airfoil in the form of an arc with a sink modeling the external-flow removal. We mean flow removal with a noticeable flow rate, which directly participates in the formation of the ambient flow, rather than boundary-layer suction, which affects the flow pattern only by changing separation conditions. The study is performed within the framework of the classical model of a steady ideal incompressible liquid flow.

Flow Pattern and Governing Equations. We consider a plane steady potential flow of an ideal incompressible liquid around an arc AB, such that the ends of the arc A and B are points of flow branching and removal, respectively, i.e., the flow is attached. On the upper surface of the arc, at the point M, there is a point sink with a flow rate $2\pi q$, and there is a critical point N between the points M and B (Fig. 1a). It is necessary to determine the most beneficial position of the sink (point M) at the arc of a given shape from the viewpoint of increasing lift force, to clarify the character of the influence of sink power on the lift force, and also to find the shape of an arc with a sink, which possesses the maximum lift force in the case of a limited chord length and curvature.

Let z be a complex variable of the flow plane. We assume that the flow velocity at infinity V_∞ is known and directed horizontally. The angle of attack of the arc (slope) is not defined beforehand and should provide the flow in accordance with the above scheme.

The flow region shown in Fig. 1a may be conformally mapped as a unit circle in the plane of an auxiliary variable ζ. After mapping, the flow around the arc is transformed into the corresponding flow around a circle (Fig. 1b). Circulation of velocity around the airfoil remains the same, and the flow velocity at infinity in the plane ζ [$z = f(\zeta)$ is a function of inverse mapping] is assumed to be equal to $U_\infty = V_\infty h$, where $h = |f'(\infty)|$. The velocity direction at infinity is determined by the relation $\mu = -\arg f'(\infty) = -\arg U_\infty$.

Institute of Mathematics and Mechanics, Kazan' State University, Kazan' 420008. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 42, No. 4, pp. 68–76, July–August, 2001. Original article submitted September 11, 2000; revision submitted January 24, 2001.

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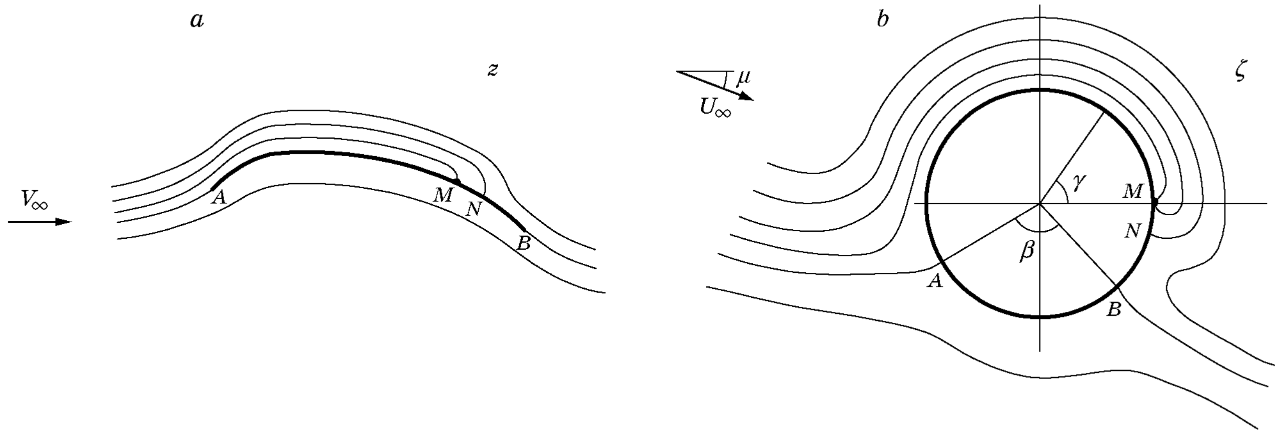

Fig. 1

In accordance with the Joukowski theorem, the lift force of an arbitrary airfoil, including that with a sink, is determined by the formula

$$Y = 2\pi\rho V_\infty \Gamma, \tag{1}$$

where Γ is the circulation of velocity divided by 2π.

The complex potential of the flow around a circle with a sink (Fig. 1b) has the following form (see, for instance, [7, p. 123]):

$$w(\zeta) = U_\infty\Big(\zeta\,\mathrm{e}^{i\mu} + \frac{1}{\zeta\,\mathrm{e}^{i\mu}}\Big) + i\Gamma\,\ln\,\zeta - q\,\ln\,\frac{(\zeta-1)^2}{\zeta} + c$$

(c is a constant). Then, the circumferential velocity is found by the formula

$$u(\gamma) = -2U_\infty \sin(\gamma+\mu) - \Gamma - q\cot(\gamma/2), \tag{2}$$

where γ is the argument (polar angle) of points on the circle; the positive direction is the direction of the velocity u that coincides with the increase in γ. The branching point A and the point of flow removal B are found from the relations

$$u(\gamma_{\rm A}) = 0, \qquad u(\gamma_{\rm B}) = 0. \tag{3}$$

We denote $\beta = \gamma_{\rm B} - \gamma_{\rm A}$. With flow parameters changing in the plane z, the values of h and β remain unchanged, since they are determined by the flow region geometry. Hence, to clarify the effect of the location and power of the sink on the lift force of a given arc, one has to determine the dependence of Γ on the corresponding parameters of the flow around a circle with a fixed value of $\beta = \beta_a$ known from the mapping $f(z)$. In finding the optimal shape, the dependence of $\beta_{\rm A}$ and h on the arc shape should be taken into account. With regard for Eqs. (2) and (3), the parameters Γ, q, and μ are related by

$$\gamma_{\rm B} - \gamma_{\rm A} = \beta. \tag{4}$$

However, it is not possible to obtain an explicit relation for the lift force in terms of the initial parameters in order to use then standard methods for studying explicit functions. Therefore, qualitative methods are used below.

The following geometric notation is used here. In the coordinates γ and u, we introduce sinusoid S and cotangent curve G described by the equations $u_S(\gamma) = -2U_0\sin(\gamma+\mu) - \Gamma$ and $u_G(\gamma) = q\cot(\gamma/2)$, respectively (Fig. 2). Note that $S = S(\mu,\Gamma)$ depends on the parameters μ and Γ [and also on h for a changing arc: $S = S(\mu,\Gamma,h)$], and $G = G(q)$ depends only on q. The quantity $u(\gamma) = u_S(\gamma) - u_G(\gamma)$ is the distance from G to S. In particular, the points A, B, and N correspond to the points of intersection of S and G. The lower surface of the arc corresponds to the section $\gamma_{\rm A} < \gamma < \gamma_{\rm B}$, where S is located above G. In this notation, the change in μ may be represented as a horizontal shift of the curve S, the change in Γ as a vertical shift, and the change in q as a vertical compression–extension of G.

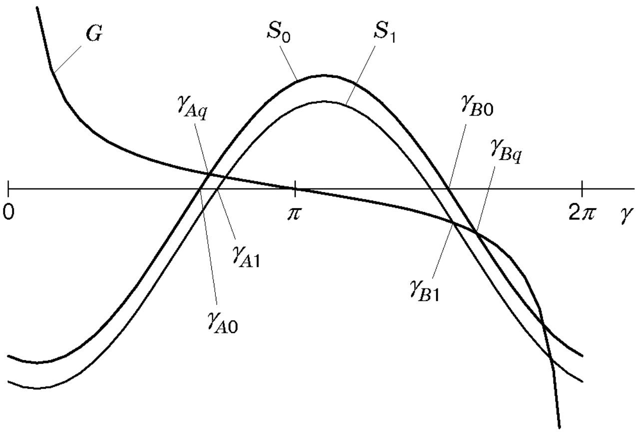

Fig. 2

Possibility of Increasing Lift Force. We show that the appearance of a sink on the arc may increase the lift force. According to Eq. (1), we have to show that the value of Γ in the presence of a sink may be greater than without it.

The absence of the sink ($q = 0$) means that G coincides with the horizontal coordinate axis within the interval $0 < \gamma < 2\pi$. We choose μ (i.e., the point of the intended sink) so that the following inequality is valid for $q = 0$:

$$\gamma_{\rm B0} - \pi \geqslant k(\pi - \gamma_{\rm A0}) \qquad (k > 1). \tag{5}$$

According to Eq. (4), we have $\gamma_{\rm B0} - \gamma_{\rm A0} = \beta_{\rm A}$. Hereinafter, the subscript "zero" corresponds to the case $q = 0$. It is allowed in (5) that $\pi - \gamma_{\rm A0} < 0$. The corresponding value of Γ is denoted as Γ_0, and $S_0 = S(\mu, \Gamma_0)$.

There is a value of q such that $\beta_q = \gamma_{\rm B\mathit{q}} - \gamma_{\rm A\mathit{q}} > \beta_{\rm A}$, where $\gamma_{\rm B\mathit{q}}$ and $\gamma_{\rm A\mathit{q}}$ are the points of intersection of S_0 and $G(q)$ [here we do not require condition (4) to be satisfied]. Indeed, by the property of cotangent $kq \cot\,(\gamma_{\rm A\mathit{q}}/2) < |q \cot\,(\gamma_{\rm B\mathit{q}}/2)|$, and if q is not too large, we obtain $\gamma_{\rm A\mathit{q}} - \gamma_{\rm A0} < \gamma_{\rm B\mathit{q}} - \gamma_{\rm B0}$, since the angles of inclination of the tangent line to S_0 at the points $\rm A_0$ and $\rm B_0$ are identical in the absolute value. Therefore, we can construct a curve $S_1 = S(\mu, \Gamma_1)$ that satisfies condition (4): $\gamma_{\rm B1} - \gamma_{\rm A1} = \beta_{\rm A}$ [$\gamma_{\rm B1}$ and $\gamma_{\rm A1}$ are the points of intersection of S_1 and $G(q)$] for $\Gamma_1 > \Gamma_0$ (i.e., S_1 lies below S_0 for an identical μ). Thus, by introducing a sink, we can increase Γ and, hence, Y.

Optimal Position of the Sink. We vary the position of the sink of a fixed power. The initial solution for G and S_1 corresponds to the scheme shown in Fig. 2 for some values of μ_1 and Γ_1 [$S_1 = S(\mu_1, \Gamma_1)$]. Without changing $\Gamma = \Gamma_1$ (and without fixing β), we reduce the value of μ by $\Delta\mu_m = \mu_1 - \mu_2 > 0$, i.e., we shift S to the right from S_1 to S_m (we use the subscript m for this solution). Because of the opposite inclination of the tangent line to S at the points A and B and owing to the monotonic character of the cotangent curve G, we obtain $\gamma_{\rm B\mathit{m}} - \gamma_{\rm B1} > \Delta\mu_m$ and $\gamma_{\rm A\mathit{m}} - \gamma_{\rm A1} < \Delta\mu_m$. (If β is small and the inclination of S at the point A is negative, then we have $\gamma_{\rm A\mathit{m}} < \gamma_{\rm A1}$.) Hence, we obtain $\beta_m = \gamma_{\rm B\mathit{m}} - \gamma_{\rm A\mathit{m}} > \gamma_{\rm B1} - \gamma_{\rm A1} = \beta_{\rm A}$. To obtain a given $\beta = \beta_{\rm A}$, we have to increase Γ from Γ_1 to a certain value Γ_2 (decreasing S from S_1 to S_2) for the same value of $\mu = \mu_2$.

If the initial solution satisfies a condition similar to Eq. (5), the angle of attack decreases. With decreasing μ, the circulation Γ increases. This is possible until there is a sector with $u < 0$ at the interval $\gamma_{\rm B} < \gamma < 2\pi$. Hence, for given q and $\beta_{\rm A}$, the maximum of Γ for this scheme is reached when the curves S and G touch each other at $\gamma = \gamma_{\rm B}$.

Thus, to increase Γ, the sink of given intensity should be located on the upper surface as close to the trailing edge as possible, so that the "reverse stream" to the sink began from the trailing edge, when the points B and N coincide. It follows from the above reasoning that, as the critical point N approaches the trailing edge B, the circulation Γ rapidly increases. In this extreme position, the flow structure near the circle is shown in Fig. 3a, and the flow pattern near the sharp trailing edge of the airfoil is depicted in Fig. 3b.

We can assume that the introduction of a sink increases flow curvature above the airfoil, as in the case of increasing circulation without a sink. Owing to the increase in the absolute value of the angle of attack (the angle of attack is negative), the flow near the front part of the airfoil is curved upward (stronger than without the sink), and the flow near the trailing edge is additionally curved downward due to the sink. Despite the increase in Γ, the conditions on sharp edges can be retained if there is a sink. Thus, the lift force increases.

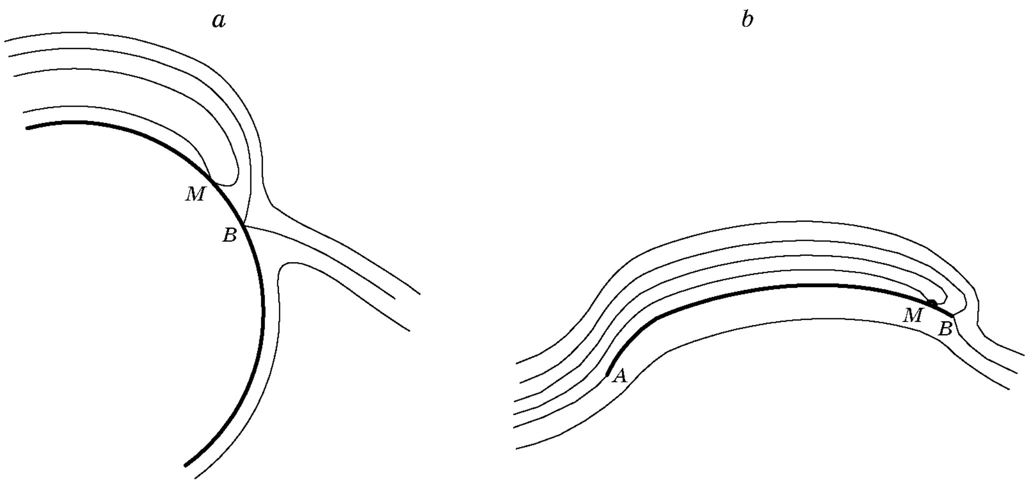

Fig. 3

It should be noted that the sink accelerates the flow on the major part of the upper surface, and the flow-branching point on the trailing edge (Fig. 3b) acts as an interceptor for the lower surface, decreasing the flow velocity and increasing the pressure. This conclusion on the most beneficial position of the sink as close to the trailing edge as possible is in agreement with the conclusions of [2–5].

We obtain the asymptotic value of the increase in the lift force ΔY with introduction of a sink for small q. Taking into account Eq. (1), we estimate the increment $\Delta\Gamma = \Gamma_e(q) - \Gamma_0$, where $\Gamma_e(q)$ is the value of Γ for the "optimal" position of the sink q on the arc. For $q = 0$, the curve G degenerates into the abscissa axis, which is crossed by the sinusoid S at the points A and B at a distance $\gamma_{\rm B} - \gamma_{\rm A} = \beta_{\rm A}$ at identical angles

$$\left|\frac{du_S}{d\gamma}\right| = 2U_\infty \sin\frac{\beta_{\rm A}}{2}.$$

For G and S touching each other and $q \to 0$, the angle γ tends to 2π and the ordinate of the touching point $u_G = u_S$ tends to zero. With accuracy to small higher orders, we have

$$u_G = \frac{2q}{2\pi - \gamma}, \qquad \frac{dU_G}{d\gamma} = -\frac{q}{2\sin^2(\gamma/2)} \approx -\frac{2q}{(2\pi-\gamma)^2};$$

at the touching point of G and S,

$$\frac{du_S}{d\gamma} = \frac{du_G}{d\gamma} \approx -2U_\infty \sin\frac{\beta_{\rm A}}{2}.$$

From here, we obtain $q/(2\pi - \gamma_{\rm B}) = U_\infty \sin(\beta_a/2)$ and $u_G(\gamma_B) \approx 2\sqrt{qU_\infty \sin(\beta_a/2)}$. For the intersection points A and B of S and G to remain at the distance $\gamma_{\rm B} - \gamma_{\rm A} = \beta_{\rm A}$, for $q \neq 0$, the curve S should be located lower than for $q = 0$, by the value $\Delta\Gamma \approx u_G(\gamma_B)/2$. Hence, the asymptotic representation of $\Delta\Gamma$ for small q has the form

$$\Delta\Gamma \approx \sqrt{q}\sqrt{U_\infty \sin(\beta_{\rm A}/2)}, \tag{6}$$

where $U_\infty = V_\infty h$.

With accuracy to small higher orders, the coefficient at $\sqrt{q}$ is independent of the small curvature of the arc, i.e., the "classical" lift force is supplemented by a "sink" lift force independent of it. The "classical" lift force at small q is also independent of q, which follows from the construction of S and G. Therefore, the contributions of the arc curvature and the sink to the lift force may be considered as independent in a certain sense.

It can be easily found that an increase in the sink power q leads to an increase in the circulation Γ only up to a certain value; then the circulation decreases. For each arc, there is an optimal value of q_* and the sink position μ_*, for which Γ reaches the maximum Γ_*.

Example of Calculation. As an example, we consider an airfoil in the form of a plate, which allows obtaining the "purely sink" lift force. For the plate, we set $\beta = \pi$ and $h = L/4$ (L is the plate length). Figure 4 shows the calculated dependences of the relative circulation $\Gamma^* = \Gamma/U_\infty = 4\Gamma/(V_\infty L)$ and the dependences $\alpha^* = \alpha/\pi$ and $\theta^* = \theta_M/\pi$ (α is the angle of attack of the plate and $\theta_M = \gamma_M - \gamma_{\rm B}$ is the arc on the circle from the image

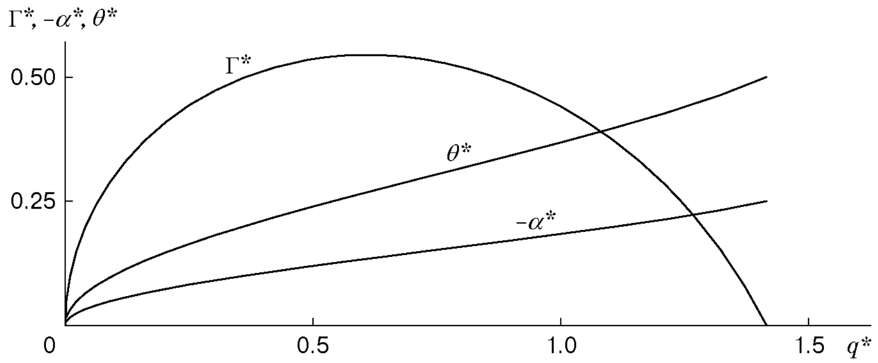

Fig. 4

of the trailing edge to the point of the sink) on $q^* = q/U_\infty = 4q/(V_\infty L)$. The curve $\Gamma^*(q^*)$ corresponds to the asymptotic curve (6) near the origin and has a maximum; reaching the maximum, this curve decreases to zero. In this case, θ^* is two times greater than $|\alpha^*|$. Note that the distance l from the trailing edge to the sink on a plate of length L is determined by the formula $l/L = (1 - \cos\theta_M)/2 = \sin^2(\theta_M/2)$, i.e., it is small for moderate q.

We can expect that, for arcs with small curvature, the value of the additional lift force due to the sink is close to that calculated for the plate, i.e., comparable with the "classical" lift force of the arc. Thus, for the characteristics of the YaK-42 aircraft, assuming that the air intake to the motors is performed from the wing (sink of the scheme considered), the increment of the lift force is commensurable with the value obtained using the conventional scheme.

Optimal Shape of the Arc. The optimal airfoil is sought in a closed set of upward convex arcs with a limited chord length $L_{\max}$ and curvature $K_{\max}$. In airfoil modeling, we confine ourselves to simple schemes, assuming that the arcs are not too curved ($K_{\max} < 2/L_{\max}$) and the ends of the chord (segment of the greatest length whose ends lie on the arc) coincide with the ends of the arc. The optimal airfoil is constructed in three stages.

Stage 1. We show that the airfoil with the maximum lift force from the given set has the maximum allowable chord length $L_{\max}$. In contrast to the case of the absence of the sink, where this is obvious, a proof is needed here.

Let there be some arc l_1 with a chord length $L_1 < L_{\max}$ on which a lift force Y_1 is reached for a certain value of q. Increasing the arc l_1 by L_2/L_1 times ($L_1 < L_2 \leqslant L_{\max}$), we introduce another arc l_2. This arc also belongs to this set, because its curvature is smaller. Since the transformation of l_1 into l_2 is a transformation of similarity, we have $h_2 = h_1 L_2/L_1 > h_1$ and $\beta_2 = \beta_1 < \pi$ (the latter inequality follows from the convex form of the arcs).

We show that a value Y_2 greater than Y_1 may be obtained for the same value of q for the sink (with its place chosen correctly) on the arc l_2.

Let the flow around the arc l_1 correspond to the curves $G = G(q)$ and $S_1 = S_1(\mu_1, \Gamma_1, h_1)$ (Fig. 5). We introduce the curve $S_f = S(\mu_f, \Gamma_1, h_2)$, choosing μ_f such that S_f crosses S_1 and G for $\gamma = \gamma_{\rm B1}$. There is a second point of intersection of S_f and S_1 for $\gamma_{fs} = \gamma_{\rm B1} - \pi$ due to the equality of the periods and the symmetry of S_f and S_1 (their horizontal axes Γ_1 coincide). Taking into account that $\gamma_{\rm A1} = \gamma_{\rm B1} - \beta$ and $\beta < \pi$, we obtain $\gamma_{fs} < \gamma_{\rm A1}$. At the sector $\gamma_{fs} < \gamma < \gamma_{\rm B1}$, the curve S_f is located above S_1; at the corresponding point of intersection of S_f and G, we have $\gamma_{Af} < \gamma_{\rm A1}$, from where $\gamma_{\rm B1} - \gamma_{Af} > \beta_2$.

Hence, we can construct a curve $S_2 = S(\mu_f, \Gamma_2, h_2)$, increasing Γ (i.e., decreasing S_f), so that the equality $\gamma_{B2} - \gamma_{A2} = \beta_2$ is valid for the points of intersection of S_2 and G. Since $\Gamma_2 > \Gamma_1$, then we have $Y_2 > Y_1$ in accordance with Eq. (1). The statement is proved.

Stage 2. We show that h increases with increasing arc curvature. Let the arc l_2 be obtained from the arc l_1 by increasing curvature at a certain section. Since the chord lengths of the arcs are fixed, and arc rotation and transfer do not affect the value of h, we may assume that the ends of the arcs l_1 and l_2 coincide. Since the arcs are convex, the sectors adjacent to the ends (with an unchanged curvature) should be turned "outside" in passing from l_1 to l_2. The length of the varied sector should increase. For simplicity, the increase in curvature is understood as a change such that the least curvature of l_2 on a given sector is greater than the greatest curvature of l_1 on it.

It is known (see, for example, [8]) that, if we consider some region that is the image of the closed Jordan curve in the plane z, then we have $d = |f'(\infty)|$, where $z = f(\zeta)$ is a function conformally mapping the image of the unit circle in the plane ζ onto this region; $f(\infty) = \infty$. Here d is the transfinite diameter of this region determined

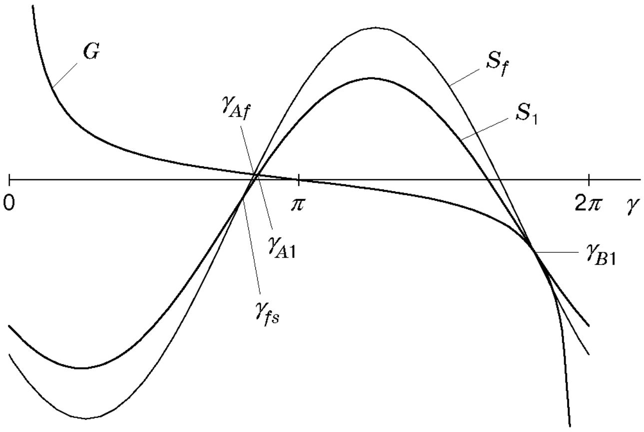

Fig. 5

by the relation

$$d = \lim_{n\to\infty}\left(\sup_{z_i,z_j} \sqrt[\frac{n(n-1)}{2}]{\prod_{i,j=1,\, i\neq j}^{n} |z_i - z_j|}\right), \tag{7}$$

where z_i and z_j are arbitrary points inside the region bounded by the curve (or on this curve).

The case considered in the present paper satisfies the above definition (h corresponds to d); therefore, Eq. (7) is also valid.

We may find a correspondence between the points of the arcs l_1 and l_2, such that $|z_{2i} - z_{2j}| \geqslant |z_{1i} - z_{1j}|$. Then, from Eq. (7), it follows that $d_2 \geqslant d_1$, i.e.,

$$h_2 \geqslant h_1 \tag{8}$$

(apparently, a strict inequality $h_2 > h_1$ may be established, but this is not needed in the case considered).

The sequence of the proof was proposed by S. R. Nasyrov.

Stage 3. We show that Y increases with increasing arc curvature. It is known [1] that the following inequality is valid for arcs obtained by "pressing out" one from another, by virtue of the Lindelöf principle [9]:

$$\beta_2 < \beta_1. \tag{9}$$

This allows us to use the same proving as at Stage 1. From the strict inequality in (9) (instead of the equality $\beta_2 = \beta_1$ at Stage 1), it follows that $Y_2 > Y_1$. The strict inequality in (8) need not be satisfied.

Thus, in the set considered, the arc with the maximum chord length and a constant maximum curvature (i.e., the arc of a circle) has the greatest lift force for a given q and, hence, in the absence of restrictions on q.

This result coincides with that obtained by Lavrent'ev for an arc without a sink [1]. Note that Lavrent'ev [1] considered a more complicated case of a set of arcs of limited length (rather than a limited chord length) and limited curvature (without requirement of convexity).

Sink on the Lower Surface. The sink on the lower surface is characterized by a constant direction (from A to B) and a limited velocity at the upper surface. Using the same approaches as in the case of a sink on the upper surface, we can easily prove that the sink on the lower surface may also increase the circulation Γ if it is located near the nose. In this case, the angle of attack α increases.

The action of the sink may be explained as follows. The rear part of the airfoil works mainly in the same manner as without the sink. The mechanism of the formation of circulation, which ensures satisfaction of the Kutta–Joukowski–Chaplygin condition, acts in this part of the airfoil. Therefore, a mere increase in the angle of attack leads to an increase in the circulation Γ. However, the mere increase in the angle of attack would violate the conditions of inflow on the leading edge, where the branching point would be shifted to the lower surface, but suction corrects the inflow on the leading edge. The necessary upward curvature of the flow upstream of the airfoil, flow acceleration on the upper surface, and a decrease in velocity (behind the sink) on the lower surface are ensured. Nevertheless, an attempt of seeking the best position of the sink in this case does not lead to a solution corresponding to the flow scheme around the airfoil.

Finally, we note that many questions remain open in studying the flow around arcs with flow removal, in particular, the study of other schemes of flow removal. For example, we can use a plate with a zero angle of attack with two symmetrically located sinks in its front and rear parts. Such a scheme yields a lift force $\sqrt{2}$ times greater than a single sink (with an identical total power of the sinks).

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 99-01-00365).

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